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APPROXIMATION OF FUNCTION IN BESOV SPACE USING EULER HAUSDORFF PRODUCT MEANS

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Abstract

In this paper, we study the degree of approximation of function in Besov space using Euler Hausdorff product means of Fourier Series and we also deduce some corollaries of our main result.
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1 Introduction

In the last few decades several researchers have studied the degree of approximation of function in Lipschitz class and Hölder space has been studied by [1,2,3,6,7] using different product summability means of Fourier series on Conjugate Fourier also. Besov space describes the smoothness properties of functions and contain many fundamental spaces such as Lipschitz space, Hölder space, etc. Mohanty *et al.* [4], Mohanty *et al.* [5], Nigam *et al.* [8] studied the approximation function in Besov space by various summability means of their Fourier series. In the present work , we obtain the degree of approximation of function in Besov space using Euler Hausdorff product means.

2 Definitions and Notations

Let $C_{2\pi} = C[0, 2\pi]$ denotes the Banach space of all 2π- periodic continuous functions f defined on $[0, 2\pi]$ under the sup norm, and

$$L_p = L_p[0, 2\pi] = \{f : [0, 2\pi] \to \mathbf{R}; \int_0^{2\pi} |f(x)|^p dx < \infty\}, p \geq 1,$$

be the space of all 2π- periodic integrable functions. The L_p- norm of function f is defined by

$$\|f\|_p := \begin{cases} \left(\frac{1}{2\pi}\int_0^{2\pi} |f(x)|^p dx\right)^{\frac{1}{p}}, & 1 \leq p < \infty \\ ess\sup_{0<x\leq 2\pi} |f(x)|, & p = \infty. \end{cases}$$

The k^{th} order modulus of smoothness of signal $f \in L_p, 0 < p \leq \infty$ is defined by

$$\omega_k(f,t)_p = \sup_{0<h\leq t} \left\|\nabla_h^k(f,\cdot)\right\|_{p'}$$

where $\delta_h^k(f,x) = \sum_{i=0}^k (-1)^{k-i}\binom{k}{i} f(x+ih), k \in \mathbf{N}$. For $p = \infty, k = 1$ and a continuous function f, the modulus of smoothness $\omega_k(f,t)_p$ reduces to the modulus of continuity $\omega(f,t)$ also for $0 < p < \infty$ and $k = 1$ $\omega_k(f,t)_p$ becomes the integral modulud of continuity $\omega(f,t)_p$.

2.1 Lipschitz Space

If a function $f \in C_{2\pi}$ and $\omega(f,t) = O(t^\alpha), 0 < \alpha \leq 1$ then $f \in Lip\ \ \alpha$. If a function $f \in L_p$, $0 < p < \infty$ and $\omega(f,t)_p = O(t^\alpha)$, $0 < \alpha \leq 1$ then $f \in Lip(\alpha, p)$. For $p = \infty$, the *class* $Lip(\alpha, p)$ reduces to the class $Lip\ \ \alpha$.

Let $\alpha > 0$ be given and let k denote the smallest integer $k > \alpha$ that is, $k = [\alpha] + 1$. For $f \in L_p$, if $\omega_k(f,t)_p = O(t^\alpha), t > 0$. Then the seminorm is

$$|f|_{Lip^*(\alpha,p)} = \sup_{t>0}(t^\alpha \omega_k(f,t)_p)$$

Thus $Lip(\alpha, p) \subseteq Lip^*(\alpha, p)$.

2.2 Hölder Space

For $0 < \alpha \leq p$, let $H_\alpha = \{f \in C_{2\pi} : \omega(f,t)\} = O(t^\alpha)$. It is known that H_α is a Banach space with norm

$$\|f\|_\alpha = \|f\|_c + \sup_{t>0}(t^{-\alpha}\omega(t)), \text{ and } \|f\|_0 = \|f\|_c$$

and $H_\alpha \subseteq H_\beta \subseteq C_{2\pi}$ for $0 < \beta \leq \alpha \leq 1$. The metric induced by the norm $\|\cdot\|_\alpha$ on H_α is call the Hölder metric.

For $0 < \alpha \leq 1$ and $0 < p \leq \infty$, let

$$H_{\alpha,p} := H_{\alpha,p}[0, 2\pi] = \{f \in C_{2\pi} : \omega(f,t)_p = \circ(t^\alpha)\}$$

with the norm $\|\cdot\|_{\alpha,p}$ defined as follows:

$$\|f\|_{\alpha,p} = \|f\|_p + \sup_{t>0}(t^{-\alpha}\omega(f,t)_p), \text{ for } 0 < \alpha \leq 1 \text{ and } \|f\|_{0,p} = \|f\|_p$$

then $H_{\alpha,p}$ is a Banach space for $p \geq 1$ and a complete p-normed space for $0 < p < 1$.

For

$$H_{\alpha,p} \subseteq H_{\beta,p} \subseteq L_p, \text{ for } 0 < \beta \leq \alpha \leq 1.$$

2.3 Besov Space

Let $\alpha > 0$ be given, and let $k = [\alpha] + 1$. For $0 < p, q \leq \infty$, the Besov space $B_q^\alpha(L_p)$ is the collection of all the 2π- periodic function $f \in L_p$ such that

$$|f|_{B_q^\alpha(L_p)} := \|\omega_k(f,\cdot)\|_{\alpha,q} = \begin{cases} \left(\int_0^\pi [t^{-\alpha}\omega(f,t)_p]^q \frac{dt}{t}\right)^{\frac{1}{q}}, & 0 < q < \infty \\ \sup_{t>0}(t^{-\alpha}\omega(f,t)_p), & q = \infty \end{cases}$$

is finite. It is known that above relation is a semi-norm if $1 \leq p, q \leq \infty$, and a quasi-norm in other case. The quasi-norm for $B_q^\alpha(L_p)$ is

$$\|f\|_{B_q^\alpha(L_p)} := \|f\|_p + |f|_{B_q^\alpha(L_p)} = \|f\|_p + \|\omega_k(f,\cdot)\|_{\alpha,q}.$$

For $q \neq \infty$, $B_q^\alpha(L_p) = Lip^*(\alpha, p)$. When $0 < \alpha < 1$, the space $B_q^\alpha(L_p)$ reduce to $H_{\alpha,p}$ and we take $p = q = \infty$ and $0 < \alpha < 1$, the besov space reduce to the H_α.

We write through the paper

$$\varphi(x,t,u) = \begin{cases} \varphi_{x+t}(u) - \varphi_x(t), & 0 < \alpha < 1 \\ \varphi_{x+t}(u) + \varphi_{x-t}(u) - 2\varphi_x(u), & 0 \leq \alpha < 2. \end{cases}$$

Theorem 2.1. *The Hausdroff matrix summability transform of $s_k(f;x)$ by $t_n^H(x)$, we get*

$$t_n^H(x) = \sum_{k=0}^{n} h_{n,k} s_k(f;x).$$

The (E,q) transform of t_n^H denoted by K_n^{EH} is given by

$$K_n^{EH} = (1+q)^{-n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \sum_{v=0}^{k} h_{n,k} s_k(f;x)$$

and

$$M_n(u) = \frac{(1+q)^{-n}}{2\pi} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \sum_{v=0}^{k} \int_0^1 \binom{k}{v} z^v (1-z)^{k-v} d\alpha(z) \frac{\sin\left(v+\frac{1}{2}\right)u}{\sin\frac{u}{2}} du.$$

3 Main Theorem

Let f be 2π-periodic functions and Lebsesgue integrable for $0 \leq \beta < \alpha < 2$. The best error approximation of f in the Besov space $B_q^\alpha(L_p)$ $p \geq 1, 1 < q \leq \infty$ by K_n^{EH} transform of its Fourier series is given by

$$E_n(f) = \left\|T_n^{EH}(\cdot)\right\|_{B_q^\alpha(L_p)} = O(1) \begin{cases} (n+1)^{-1}, & \alpha - \beta - q^{-1} > 1 \\ (n+1)^{-\alpha+\beta+q^{-1}}, & \alpha - \beta - q^{-1} < 1 \\ (n+1)^{-1}[\log(n+1)\pi]^{1-q^{-1}}, & \alpha - \beta - q^{-1} = 1. \end{cases}$$

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4 Lemmas

We need following lemmas in the proof of our main result.

Lemma 4.1. *([1])* $|M_n(u)| = O(n+1)$, *for* $0 \le u \le \frac{1}{(n+1)}$.

Lemma 4.2. *([1])* $|M_n(u)| = O((n+1)^{-1}u^{-2})$ *for* $\frac{1}{(n+1)} \le u \le \pi$.

Lemma 4.3. *([4]) Let* $1 \le p \le \infty$, *and* $0 < \alpha < 2$. *If* $f \in L_p$ *then for* $0 < t,\ u \le \pi$
(i) $\|\varphi(\cdot,t,u)\|_p \le 4\omega_k(f,t)_p$
(ii) $\|\varphi(\cdot,t,u)\|_p \le 4\omega_k(f,u)_p$,
(iii) $\|\varphi(u)\| \le 2\omega_k(f,u)_p$,
where $k = [\alpha]+1$.

Lemma 4.4. *([4]) Let* $0 < \beta < \alpha < 2$. *If* $f \in B_q^\alpha(L_p)$, $p \ge 1, 1 < 1 < \infty$, *then*

$$\int_0^\pi |M_n(u)| \left(\int_0^u \frac{\|\varphi(\cdot,t,u)\|_p^q}{t^{\beta q}}\frac{dt}{t}\right)^{\frac{1}{q}} = O(1)\left\{\int_0^\pi (u^{\alpha-\beta}|M_n(u)|)^{\frac{q}{q-1}}du\right\}^{1-\frac{1}{q}}$$

$$= O(1)\left\{\int_0^\pi \left(u^{\alpha-\beta+\frac{1}{q}}|M_n(u)|\right)^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}}.$$

Lemma 4.5. *([4]) Let* $0 \le \beta < \alpha < 2$ *and* $f \in B_q^\alpha(L_p), p \ge 1, q = \infty$ *then*

$$\sup_{0<t,u\le\pi} (t^{-\beta}\|\varphi(\cdot,t,u)\|_p) = O(u^{\alpha-\beta}).$$

Lemma 4.6. *(i)* $N_n(y,t) = \int_0^\pi M_n(u)\phi(y,t,u)$,
(ii) $\omega_k(T_r,t) = \|N_n(\cdot,t)\|_p$.

5 Proof of the Main theorem

5.1 Case I:

For $1 < q < \infty$, $p \ge 1$, $0 \le \beta < \alpha < 2$.

Proof. We have

$$s_k(f;x) - f(x) = \frac{1}{2\pi}\int_0^\pi \varphi(x,t)\frac{\sin\left(k+\frac{1}{2}\right)t}{\sin\frac{t}{2}}dt.$$

The Hausdorff matrix summability transform of $s_k(f;x)$ by $t_n^H(x)$, we get

$$\begin{aligned} t_n^H(x) - f(x) &= \sum_{k=0}^n h_{n,k}\{s_k(f;x)-f(x)\} \\ &= \frac{1}{2\pi}\varphi(x,t)\sum_{k=0}^n \binom{n}{k}\Delta^{n-k}\left(\int_0^1 z^k d\alpha(z)\right)\frac{\sin\left(k+\frac{1}{2}\right)t}{\sin\frac{t}{2}}dt \\ &= \frac{1}{2\pi}\varphi(x,t)\sum_{k=0}^n \int_0^1 \binom{n}{k} z^k(1-z)^{n-k}d\alpha(z)\frac{\sin\left(k+\frac{1}{2}\right)t}{\sin\frac{t}{2}}dt. \end{aligned} \tag{5.1}$$

The (E,q) transform of t_n^H denoted by K_n^{EH} is given by

$$\begin{aligned} K_n^{EH} - f(x) &= (1+q)^{-n}\sum_{k=0}^n \binom{n}{k} q^{n-k}\{t_n^H(x)-f(x)\} \\ &= (1+q)^{-n}\sum_{k=0}^n \binom{n}{k} q^{n-k}\left\{\frac{1}{2\pi}\int_0^\pi \varphi(x,t)\sum_{v=0}^k\int_0^1\binom{k}{v}z^v(1-z)^{k-v}d\alpha(z)\frac{\sin\left(v+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)}dt\right\} \\ &= (1+q)^{-n}\sum_{k=0}^n \binom{n}{k} q^{n-k}\left\{\frac{1}{2\pi}\int_0^\pi \varphi(x,u)\sum_{v=0}^k\int_0^1\binom{k}{v}z^v(1-z)^{k-v}d\alpha(z)\frac{\sin\left(v+\frac{1}{2}\right)u}{\sin\left(\frac{u}{2}\right)}du\right\} \end{aligned}$$

(Replacing t by u)

$$= \int_0^\pi \varphi_x(u) M_n(u) dt. \tag{5.2}$$

Let

$$T_n(x) = K_n^{EH}(x) - f(x) = \int_0^\pi \varphi(x,u) M_n(u) dt. \tag{5.3}$$

Using the definition of Besov norm, we have

$$\|f\|_{B_q^\alpha(L_p)} := \|f\|_p + |f|_{B_q^\alpha(L_p)} = \|f\|_p + \|\omega_k(f,\cdot)\|_{\alpha,q}.$$

$$\|T_n(\cdot)\|_{B_q^\beta(L_p)} = \|T_n(\cdot)\|_p + \|\omega_k(T_n,\cdot)\|_{\beta,q}. \tag{5.4}$$

Using Lemma 4.3(iii), we get

$$\|T_n(\cdot)\|_p \leq \int_0^\pi \|\varphi(u)\|_p |M_n(u)| du \leq \int_0^\pi 2\omega_k(f,u)_p |M_n(u)| du. \tag{5.5}$$

□

Employing Hölder inequality, we have

$$\|T_n(\cdot)\|_p \leq 2\left\{\int_0^\pi \left(u^{\alpha+\frac{1}{q}}|M_n(u)|\right)^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}} \left\{\int_0^\pi \left(\frac{\omega_k(f,u)_p}{u^{\alpha+\frac{1}{q}}}\right)^q du\right\}^{\frac{1}{q}}.$$

Making an appeal to Besov space definition , we establish

$$\|T_n(\cdot)\|_p = O(1)\left\{\int_0^\pi \left(u^{\alpha+\frac{1}{q}}|M_n(u)|\right)^{\frac{q}{q-1}}\right\}^{1-\frac{1}{q}}$$

$$= O(1)\left[O(1)\left\{\int_0^{\frac{1}{n+1}} \left(u^{\alpha+\frac{1}{q}}|M_n(u)|\right)^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}} + O(1)\left\{\left\{\int_{\frac{1}{n+1}}^\pi \left(u^{\alpha+\frac{1}{q}}|M_n(u)|\right)^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}}\right\}\right]$$

$$= O(1)(E+H). \tag{5.6}$$

Using Lemma 4.1 in E of (5.6), we attain

$$E = \left\{\int_0^{\frac{1}{n+1}} \left(u^{\alpha+\frac{1}{q}}|M_n(u)|\right)^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}}$$

$$O\left\{\int_0^{\frac{1}{n+1}} \left(u^{\alpha+\frac{1}{q}}(n+1)\right)^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}}$$

$$= \left\{(n+1)^{\frac{q}{q-1}} \int_0^{\frac{1}{n+1}} \left(u^{\alpha+\frac{1}{q}}\right) du\right\}^{1-\frac{1}{q}}$$

$$= O(n+1)^{-\alpha}. \tag{5.7}$$

Emplying Lemma 4.2 in H of (5.6), we derive

$$H = \left\{\int_{\frac{1}{n+1}}^\pi \left(u^{\alpha+\frac{1}{q}}|M_n(u)|\right)^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}}$$

$$= \left\{\int_{\frac{1}{n+1}}^\pi \left(u^{\alpha+\frac{1}{q}}\frac{1}{(n+1)u^2}\right)^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}}$$

$$= \left\{\int_{\frac{1}{n+1}}^\pi \left(u^{\alpha+\frac{1}{q}-2}\frac{1}{(n+1)}\right)^{\frac{q}{q-1}} du\right\}^{1-\frac{1}{q}}$$

$$= O(n+1)^{-1}\left\{\int_{\frac{1}{n+1}}^\pi u^{\frac{q}{q-1}(\alpha-1)-1} du\right\}^{1-\frac{1}{q}}$$

$$O(1)\begin{cases} (n+1)^{-1}, & \alpha > 1 \\ (n+1)^{-\alpha}, & \alpha < 1 \\ (n+1)^{-1}[\log(n+1)\pi]^{1-q^{-1}} & \alpha = 1. \end{cases} \tag{5.8}$$

So, we get

$$\|T_n(\cdot)\|_p = O(1)\begin{cases}(n+1)^{-1}, & \alpha>1\\ (n+1)^{-\alpha}, & \alpha<1\\ (n+1)^{-1}[\log(n+1)\pi]^{1-q^{-1}} & \alpha=1.\end{cases} \tag{5.9}$$

By using generalized Minkowskis inequality and Lemma 4.4, we have

$$\begin{aligned}\|\omega_k(T_n,\cdot)\| &= \left\{\int_0^\pi\left(\frac{\omega_k(T_n,t)_p}{t^\beta}\right)^q\frac{dt}{t}\right\}^{\frac{1}{q}}\\ &= \left\{\int_0^\pi\left(\frac{\|N_n(\cdot,t)\|_p}{t^\beta}\right)^q\frac{dt}{t}\right\}^{\frac{1}{q}}\\ &= \int_0^\pi|M_n(u)|du\left\{\int_0^u\frac{\|\varphi(\cdot,t,u)\|_p^q}{t^{\beta q}}\frac{dt}{t}\right\}^{\frac{1}{q}} + \int_0^\pi|M_n(u)|du\left\{\int_u^\pi\frac{\|\varphi(\cdot,t,u)\|_p^q}{t^{\beta q}}\frac{dt}{t}\right\}^{\frac{1}{q}}\\ &= O(1)\left\{\int_0^\pi(u^{\alpha-\beta}|M_n(u)|^{\frac{q}{q-1}})du\right\}^{1-\frac{1}{q}} + O(1)\left\{\int_0^\pi(u^{\alpha-\beta+\frac{1}{q}}|M_n(u)|^{\frac{q}{q-1}})du\right\}^{1-\frac{1}{q}}\\ &= O(1)(E_1+H_1).\end{aligned} \tag{5.10}$$

Now, $(a+b)^r \le a^r+b^r$ for positive a,b and $0<r\le 1$ for $r=1-\frac{1}{q}<1$. we have

$$\begin{aligned}E_1 &= \left\{\int_0^\pi(u^{\alpha-\beta}|M_n(u)|^{\frac{q}{q-1}})du\right\}^{1-\frac{1}{q}}\\ &\le \left\{\left(\int_0^{\frac{1}{n+1}}+\int_{\frac{1}{n+1}}^\pi\right)(u^{\alpha-\beta}|M_n(u)|)^{\frac{q}{q-1}}\right\}^{1-\frac{1}{q}}\\ &= E_{11}+E_{12}.\end{aligned} \tag{5.11}$$

Using Lemma 4.1, we have

$$\begin{aligned}E_{11} &= \left\{\int_0^{\frac{1}{n+1}}(u^{\alpha-\beta}|M_n(u)|)^{\frac{q}{q-1}}du\right\}^{1-\frac{1}{q}}\\ &= \left\{\int_0^{\frac{1}{n+1}}(u^{\alpha-\beta}(n+1))^{\frac{q}{q-1}}du\right\}^{1-\frac{1}{q}}\\ &= O\left\{(n+1)^{-\alpha+\beta+\frac{1}{q}}\right\}.\end{aligned} \tag{5.12}$$

Using Lemma 4.2 in E_{12}, we have

$$\begin{aligned}E_{12} &= \left\{\int_{\frac{1}{n+1}}^\pi(u^{\alpha-\beta}|M_n(u)|)^{\frac{q}{q-1}}du\right\}^{1-\frac{1}{q}}\\ &= \left\{\int_{\frac{1}{n+1}}^\pi\left(u^{\alpha-\beta}\frac{1}{(n+1)u^2}\right)^{\frac{q}{q-1}}du\right\}^{1-\frac{1}{q}}\\ &= O(1)\begin{cases}(n+1)^{-1}, & \alpha-\beta-\frac{1}{q}>1,\\ (n+1)^{-\alpha+\beta+\frac{1}{q}}, & \alpha-\beta-\frac{1}{q}<1,\\ (n+1)^{-1}\log[(n+1)\pi]^{1-q^{-1}} & \alpha-\beta-\frac{1}{q}=1.\end{cases}\end{aligned} \tag{5.13}$$

Combining (5.11)(5.12) and (5.13), we establish

$$E_1 = O(1)\begin{cases}(n+1)^{-1}, & \alpha-\beta-\frac{1}{q}>1,\\ (n+1)^{-\alpha+\beta+\frac{1}{q}}, & \alpha-\beta-\frac{1}{q}<1,\\ (n+1)^{-1}\log[(n+1)\pi]^{1-q^{-1}} & \alpha-\beta-\frac{1}{q}=1.\end{cases} \tag{5.14}$$

Now,

$$H_1 = \left\{ \int_0^{\pi} \left(u^{\alpha-\beta+\frac{1}{q}} |M_n(u)| \right)^{\frac{q}{q-1}} \right\}^{1-\frac{1}{q}}$$
$$\leq \left\{ \left(\int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\pi} \right) \left(u^{\alpha-\beta+\frac{1}{q}} |M_n(u)| \right)^{\frac{q}{q-1}} \right\}^{1-\frac{1}{q}}$$
$$= H_{11} + H_{12}. \tag{5.15}$$

Using Lemma 4.1 in H_{11}, we derive

$$H_{11} = \left\{ \int_0^{\frac{1}{n+1}} \left(u^{\alpha-\beta+\frac{1}{q}} |M_n(u)| \right)^{\frac{q}{q-1}} du \right\}^{1-\frac{1}{q}}$$
$$= \left\{ \int_0^{\frac{1}{n+1}} \left(u^{\alpha-\beta+\frac{1}{q}} (n+1) \right)^{\frac{q}{q-1}} du \right\}^{1-\frac{1}{q}}$$
$$= O\{(n+1)^{-\alpha+\beta}\}. \tag{5.16}$$

Using Lemma 4.2 in H_{12}, we obtain

$$H_{12} = \left\{ \int_{\frac{1}{n+1}}^{\pi} \left(u^{\alpha-\beta+\frac{1}{q}} |M_n(u)| \right)^{\frac{q}{q-1}} du \right\}^{1-\frac{1}{q}}$$
$$= \left\{ \int_{\frac{1}{n+1}}^{\pi} \left(u^{\alpha-\beta+\frac{1}{q}} \frac{1}{(n+1)u^2} \right)^{\frac{q}{q-1}} du \right\}^{1-\frac{1}{q}}$$
$$= O(1) \begin{cases} (n+1)^{-1}, & \alpha-\beta > 1 \\ (n+1)^{-\alpha+\beta}, & \alpha-\beta < 1 \\ (n+1)^{-1}[\log(n+1)\pi]^{1-q^{-1}}, & \alpha-\beta = 1. \end{cases} \tag{5.17}$$

Combining (5.15), (5.16) and (5.17), we get

$$H_1 = O(1) \begin{cases} (n+1)^{-1}, & \alpha-\beta > 1 \\ (n+1)^{-\alpha+\beta}, & \alpha-\beta < 1 \\ (n+1)^{-1}[\log(n+1)\pi]^{1-q^{-1}}, & \alpha-\beta = 1. \end{cases} \tag{5.18}$$

From (5.10), (5.14) and (5.18), we obtain

$$\|\omega_k(T_n,\cdot)\|_{\beta,q} = O(1) \begin{cases} (n+1)^{-1}, & \alpha-\beta-\frac{1}{q} > 1, \\ (n+1)^{-\alpha+\beta+\frac{1}{q}}, & \alpha-\beta-\frac{1}{q} < 1, \\ (n+1)^{-1}\log[(n+1)\pi]^{1-q^{-1}} & \alpha-\beta-\frac{1}{q} = 1. \end{cases} \tag{5.19}$$

From (5.4), (5.9) and (5.19), we derive

$$\|T_n(\cdot)\|_{B_q^{\beta}(L_p)} = O(1) \begin{cases} (n+1)^{-1}, & \alpha-\beta-\frac{1}{q} > 1, \\ (n+1)^{-\alpha+\beta+\frac{1}{q}}, & \alpha-\beta-\frac{1}{q} < 1, \\ (n+1)^{-1}\log[(n+1)\pi]^{1-q^{-1}} & \alpha-\beta-\frac{1}{q} = 1. \end{cases} \tag{5.20}$$

5.2 Case II

For $q = \infty$, $0 \leq \beta < \alpha < 2$.

$$\|T_n(\cdot)\|_{B_\infty^{\beta}(L_p)} = \|T_n(\cdot)\|_p + \|\omega_k(T_n,\cdot)\|_{\beta,\infty}. \tag{5.21}$$

Using condition $\omega_k(f,t) = O(t^\alpha), t > 0$ in (5.5), we have

$$\|T_n(\cdot)\|_p = \int_0^{2\pi} 2\omega_k(f,u)|M_n(u)|du$$
$$= O(1) \left\{ \int_0^{\frac{1}{n+1}} |M_n(u)|u^\alpha du + \int_{\frac{1}{n+1}}^{\pi} |M_n(u)|u^\alpha du \right\}$$

$$= O(1)[E_2 + H_2]. \tag{5.22}$$

Applying Lemma 4.1, we have

$$\begin{aligned} E_2 &= \int_0^{\frac{1}{n+1}} |M_n(u)| u^\alpha du \\ &\le \int_0^{\frac{1}{(n+1)}} u^\alpha (n+1) du \\ &= (n+1)^{-\alpha}. \end{aligned} \tag{5.23}$$

Using Lemma 4.2, we derive

$$\begin{aligned} H_2 &= \int_{\frac{1}{n+1}}^{\pi} |M_n(u)| u^\alpha du \\ &\le \frac{1}{n+1} \int_{\frac{1}{n+1}}^{\pi} u^\alpha \frac{1}{u^2} du \\ &= \begin{cases} (n+1)^{-1}, & \alpha > 1 \\ (n+1)^{-\alpha}, & \alpha < 1 \\ (n+1)^{-1}[\log(n+1)\pi], & \alpha = 1. \end{cases} \end{aligned} \tag{5.24}$$

An appeal to (5.22), (5.23) and (5.24), gives

$$\|T_n(\cdot)\|_p = O(1) \begin{cases} (n+1)^{-1}, & \alpha > 1 \\ (n+1)^{-\alpha}, & \alpha < 1 \\ (n+1)^{-1}[\log(n+1)\pi], & \alpha = 1. \end{cases} \tag{5.25}$$

Making an appeal to generalized Minkowaskis inequality and Lemma 4.6, we derive

$$\begin{aligned} \|\omega_k(T_n, \cdot)\|_{\beta,q} &= \sup_{t>0} (t^{-\beta} \omega_k(T_n, t)_p) \\ &= \sup_{t>0} (t^{-\beta} \|N_n(\cdot, t)\|_p) \\ &= \sup_{t>0} \left[t^{-\beta} \left(\frac{1}{2\pi} \int_0^{2\pi} |M_n(u)| |\varphi(x,t,u) du|^p dx \right)^{\frac{1}{p}} \right] \\ &= \sup_{t>0} \left[t^{-\beta} \left(\frac{1}{2\pi} \right)^p \int_0^{2\pi} \{ |M_n(u)|^p |\varphi(x,t,u)|^p dx \}^{\frac{1}{p}} du \right] \\ &= \int_0^\pi \left(\sup_{t>0} t^{-\beta} \|\varphi(\cdot, t, u)\|_p \right) |M_n(u)| du \\ &= O(1) \int_0^\pi u^{\alpha-\beta} |M_n(u)| du \\ &= O(1) \left[\left(\int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\pi} \right) u^{\alpha-\beta} |M_n(u)| du \right] \\ &= O(1)(E_3 + H_3). \end{aligned} \tag{5.26}$$

Using Lemma 4.1 in E_3, we have

$$E_3 = \int_0^{\frac{1}{n+1}} u^{\alpha-\beta} |M_n(u)| du = O\{(n+1)^{\alpha-\beta}\}. \tag{5.27}$$

Making an appeal to Lemma 4.2 in H_3, we derive

$$\begin{aligned} H_3 &= \int_{\frac{1}{n+1}}^{\pi} u^{\alpha-\beta} |M_n(u)| du \\ &= O(1) \frac{1}{n+1} \int_{\frac{1}{n+1}}^{\pi} u^{\alpha-\beta-2} du \end{aligned}$$

$$= O(1)\begin{cases}(n+1)^{-1}, & \alpha-\beta>1\\(n+1)^{-\alpha-\beta}, & \alpha-\beta<1\\(n+1)^{-1}[\log(n+1)\pi], & \alpha-\beta=1.\end{cases} \tag{5.28}$$

An appeal to (5.26), (5.27) and (5.28) gives

$$\|\omega_k(T_n,\cdot)\|_{\beta,\infty} = O(1)\begin{cases}(n+1)^{-1}, & \alpha-\beta>1\\(n+1)^{-\alpha-\beta}, & \alpha-\beta<1\\(n+1)^{-1}[\log(n+1)\pi], & \alpha-\beta=1.\end{cases} \tag{5.29}$$

Emplying (5.21), (5.25) and (5.29), we establish

$$\|T_n(\cdot)\|_{B^{\beta}_{\infty}(L_p)} = O(1)\begin{cases}(n+1)^{-1}, & \alpha-\beta-\frac{1}{q}>1,\\(n+1)^{-\alpha+\beta+\frac{1}{q}}, & \alpha-\beta-\frac{1}{q}<1,\\(n+1)^{-1}\log[(n+1)\pi]^{1-q^{-1}} & \alpha-\beta-\frac{1}{q}=1.\end{cases} \tag{5.30}$$

6 Some Proposition

The following corollary can be derived from our main theorem.

Corollary 6.1. *The best Error approximation of f in the Besov space $B^{\alpha}_{q}(L_p), p \geq 1, 1 < q \leq \infty$, by $(E,q)(C,\delta)$ means of its Fourier series is given by*

$$E_n(f) = \|T_n(\cdot)\|_{B^{\alpha}_{q}(L_p)} = O(1)\begin{cases}(n+1)^{-1}, & \alpha-\beta-\frac{1}{q}>1,\\(n+1)^{-\alpha+\beta+q^{-1}}, & \alpha-\beta-\frac{1}{q}<1,\\(n+1)^{-1}\log[(n+1)\pi]^{1-q^{-1}} & \alpha-\beta-\frac{1}{q}=1.\end{cases} \tag{6.1}$$

Remark 6.1. *Corollary* 6.1 *can be further reduce in $(E,1)(C,\delta)$ means, $(E,q)(C,1)$ means and $(E,1)(C,1)$ means.*

7 Conclusion

Many researchers use various summability means to obtain the degree of approximation of functions in various spaces such as Lipschitz space, Hölder space etc. Because the Besov space generalizes to more elementary function, this space is very effective at measuring the regularity of functions. Our result generalizes many known results obtained using the Lipschitz space.

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References

[1] A. A. Das, S. K. Paikray, T. Pradhan and H. Dutta, Approximation of Signal in the Weighted Zygmund Class via Euler Hausdroff Product Summability mean of Fourier Series, *Jour. Indian Math. Soc.,* **86**(3-4) (2019), 296-314.

[2] S. Lal, Approximation of functions belonging to the generalized Lipschitz class by $C^1 \cdot N_p$ summability method of Fourier series, *Appl. Math. Comp.,* **209** (2009), 346 - 350.

[3] S. Lal and A. Mishra, , Euler- Hausdorff matrix summability operator and trigonometric approximation of the conjugate of a function belonging to the generalized Lipschitz class, *Journal of inequalities and Applications,* **14** (2013), 02 -14.

[4] H. Mohanty, G. Das and B. K. Ray, Degree of Approximation of Fourier Series of Functions in Besov Space by (N,p_n) Mean, *Journal of the Orissa Mathematical Society,* **30**(2) (2011), 13-34.

[5] M. Mohanty, G. Das and S. Beuria, Degree of Approximation of Functions by their Fourier Series in the Besov Space By Matrix Mean, *International Journal of Mathematics and its applications,* **4** (2016), 69-84.

[6] H. K. Nigam and K. Sharma, Approximation of functions belonging to different classes of functions by $(E,1)(N,p_n)$ product means, *Lobachevskii Journal of Mathematics,* **32**(4) (2011), 345 -357.

[7] H. K. Nigam, and K. Sharma, On $(E,1)(C,1)$ summability of Fourier series and its conjugate series, *Int. J. Pure Appl. Math.,* **82**(3) (2013), 365 -375.

[8] H. K. Nigam and M. Hadish, Functional approximation in Besov space using Generalized Norlund - Hausdroff product matrix, *Journal of Inequalities and Applications,* **191**(2019), pages??. https//doi.org/10.1186/s13660-019-2128-1 (2019).